A Generalized Integration Formula for Discrete – Time Simulation based on Piecewise Polynomial Signal Approximation

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International Conference on APPLIED MATHEMATICS (ICAM05)

22 – 26 August 2005, ITB Bandung, Indonesia

Contents

1. Introduction

- Signal and system in the real world can be considered to be continuous – time (CT) one defined on the time – axis.
- Discrete – time (DT) model simulating a CT system has been successfully applied to many applications, e.g. real – time DT flight simulation, by using digital computer.
- Integration is important stage in the simulation of system.
- The given problem is a CT differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \mathrm{f}(x(t), u(t), t) = s(t)$$

The CT solution of indefinite integral is defined by

$$x(t) = x(a) + \int_a^t s(\tau)\,\mathrm{d}\tau$$

1. Introduction (cont.)

- In the classical approach, the difference equations that describe discrete approximation of continuous integration are Euler's integration formula, which describes the process of sampling the continuous integration of a zero – order reconstructed integran

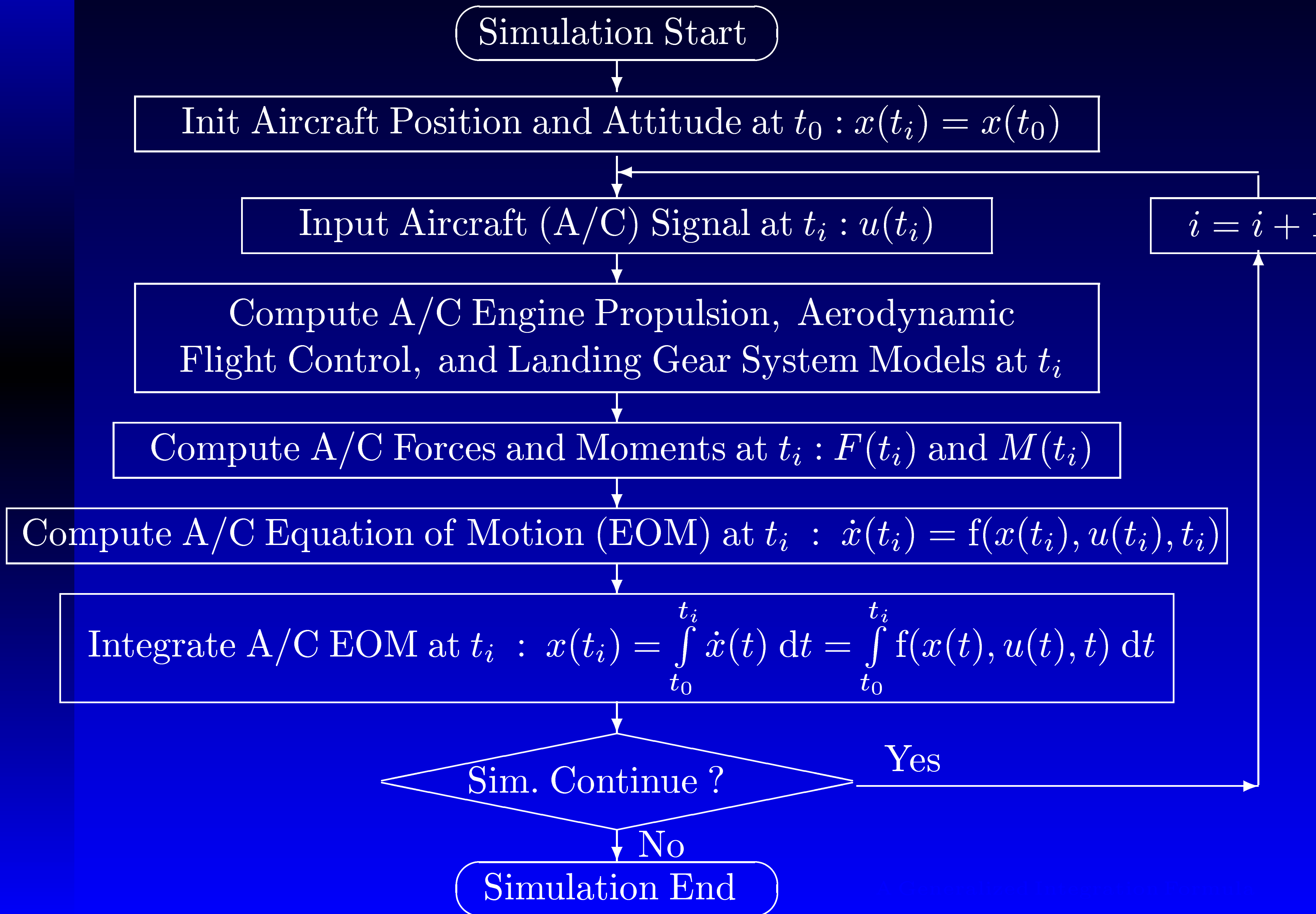
$$x_k = x_{k-1} + h\dot{x}_{k-1}$$

and trapezoidal integration formula

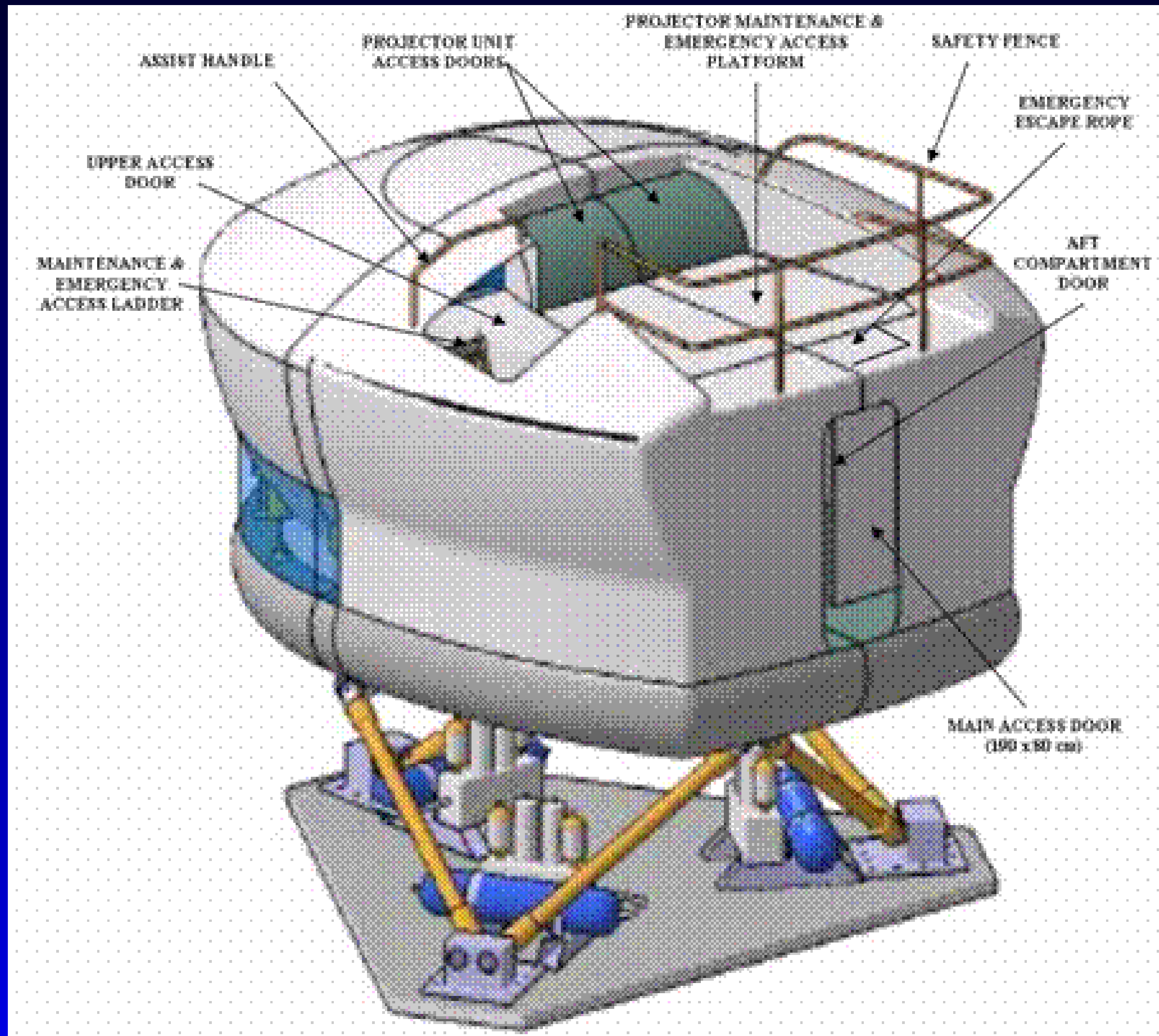
$$x_k = x_{k-1} + \frac{h}{2}(\dot{x}_k + \dot{x}_{k-1})$$

- Purpose : Derives a generalized integration method by assuming that the signal $s(t)$ is the m order fluency signal. We first have to define the signal space. Then we have a mathematically strict relationship between the CT signal and the DT signal. The relationship gives us a DT integration model approximating a CT one with a clear meaning.

Role of Integration in Simulation



Example of Training Flight Simulator



2. Mathematical Preliminaries

- The fluency signal of order m is an $(m-2)$ −times continuously differentiable piecewise polynomial of degree $m-1$, and it is identical with a staircase or polygonal signal when its order is 1 or 2 respectivelly. A fluency signal approaches a band − limited signal when its order approaches infinity as its smoothness increases in proportion to its order.
- Define the signal space composed of fluency signals of order m (for $m = 1, 2, \ldots$) :

$$ {}^{m}S := [{}^{m}_{[s]}\psi_k(t)]_{k=-\infty}^{\infty} $$

- $\{{}^{m}_{[s]}\psi_k(t)\}_{k=-\infty}^{\infty}$ is a fluency sampling basis :

$$ {}^{m}_{[s]}\psi_k(t) := \sum_{\ell=-\infty}^{\infty} {}^{m}\beta(\ell-k)\, {}^{m}_{[b]}\psi_\ell(t) \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots $$

$\{{}^{m}_{[b]}\psi_\ell(t)\}_{\ell=-\infty}^{\infty}$ is a B − spline basis of order m defined by

$$ {}^{m}_{[b]}\psi_\ell(t) := \int_{-\infty}^{\infty} \left(\frac{\sin(\pi f h)}{\pi f h}\right)^m \exp^{j2\pi(t-(\ell+\frac{m}{2})h)f}\, \mathrm{d}f $$

$$ \text{for } \ell = 0, \pm 1, \pm 2, \ldots $$

2. Mathematical Preliminaries (cont.)

- And $\{{}^{m}\beta(k)\}_{k=-\infty}^{\infty}$ is defined by

$$
\begin{aligned}
{}^{m}\beta(k) &:= h\int_{-\frac{1}{2h}}^{\frac{1}{2h}} {}^{m}_{f}B(f)\ \exp^{j2\pi fkh}\,\mathrm{d}f \\
{}^{m}_{f}B(f) &:= \frac{h}{\sum_{p=-\infty}^{\infty}\left[\frac{\sin(\pi(fh-p))}{\pi(fh-p)}\right]^{m}}
\end{aligned}
$$

- The B – spline basis can be represented in the form of a piecewise polynomial of degree $m-1$:

$$
{}^{m}_{[b]}\psi_{\ell}(t) = \frac{m}{h^{m-1}}\sum_{p=0}^{m}\frac{(-1)^{p}(t-(\ell+p)h)_{+}^{m-1}}{p!(m-p)!}
$$

which is $(m-2)$ –times continuously differentiable over the t –axis, where

$$
(t-a)_{+}^{m-1} = \begin{cases} (t-a)^{m-1}, & (t>a) \\ 0, & (t\leq a) \end{cases}
$$

2. Mathematical Preliminaries (cont.)

- A B – spline signal ${}^{m}_{[b]}\psi_\ell(t)$ satisfies :

  (a) a time – limited (locally supported) property

  $${}^{m}_{[b]}\psi_\ell(t) = 0, \quad t \notin (\ell h, (\ell + m)h)$$

  (b) a shifting property

  $${}^{m}_{[b]}\psi_\ell(t + kh) = {}^{m}_{[b]}\psi_{\ell-k}(t)$$

  (c) symmetry property

  $${}^{m}_{[b]}\psi_\ell(-t) = {}^{m}_{[b]}\psi_{-m-\ell}(t) \qquad {}^{m}_{[b]}\psi_\ell(h - t) = {}^{m}_{[b]}\psi_{-m+1-\ell}(t)$$

- The fluency sampling basis satisfies

  $$s(t) = \sum_{k=-\infty}^{\infty} s_k \, {}^{m}_{[s]}\psi_k(t)$$

  for any signal $s(t) \in {}^{m}S$, where $s_k = s(t_k)(t_k = kh + (mh/2))$, is a sampling value. We call a signal $s(t)$, formed with a fluency sampling basis with the appropriate order m, the fluency signal.

2. Mathematical Preliminaries (cont.)

- The fluency sampling basis is not given explicitly. The B – spline basis shall be used as a medium to investigate it, which is given by

$$s(t) = \sum_{\ell=-\infty}^{\infty} w_\ell \, {}^{m}_{[b]}\psi_\ell(t)$$

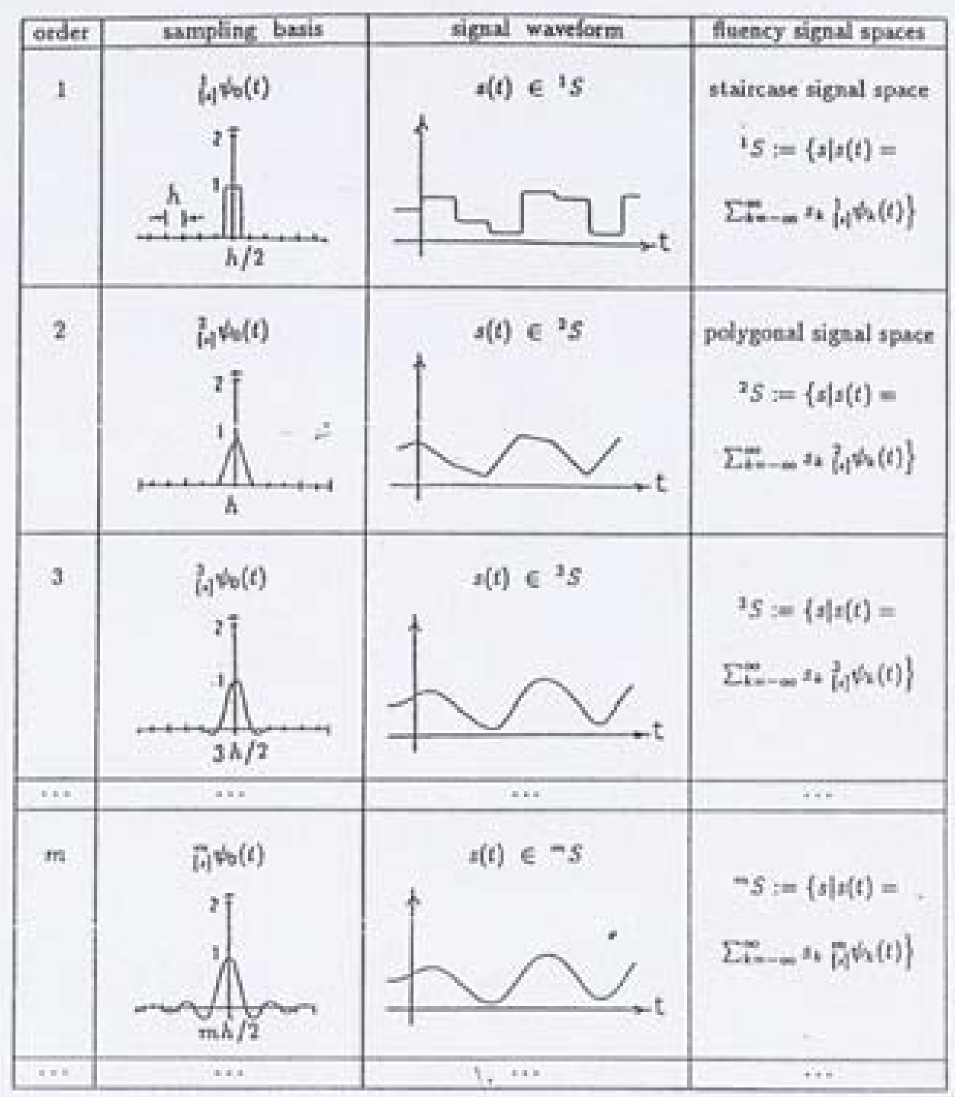
  where ${}^{m}_{[b]}\psi_\ell(t)$ means the B – spline basis and w_ℓ is the B – spline coefficient.

- ${}^{m}_{[s]}\varphi$ denote the mapping which transforms a CT signal $s(t)$ into its corresponding DT one s_k, and ${}^{m}_{[b]}\varphi$ denotes the mapping that transforms $s(t)$ into w_ℓ. Then ${}^{m}_{[s]}\varphi$ is a linear bijection on ${}^{m}S$ onto l_2, and ${}^{m}_{[b]}\varphi$ is a linear bijection on ${}^{m}S$ onto l_2. In order to represent the relation between ${}^{m}_{[s]}\varphi$ and ${}^{m}_{[b]}\varphi$, the coordinate transform operator from ${}^{m}_{[s]}\varphi$ to ${}^{m}_{[b]}\varphi$ shall be defined as :

$${}^{m}B := {}^{m}_{[b]}\varphi \, {}^{m}_{[s]}\varphi^{-1}$$

2. Mathematical Preliminaries (cont.)

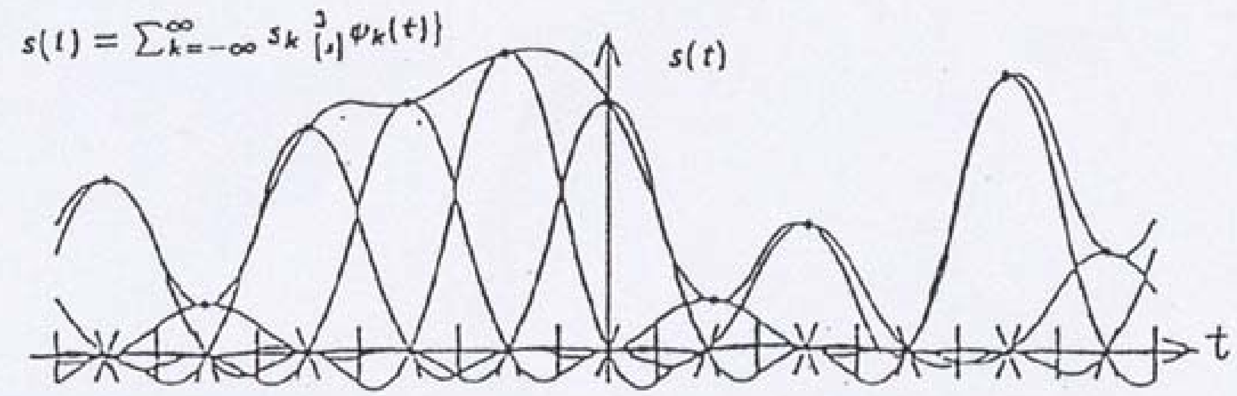
- Waveform of the $s(t)$ and the sampling basis.



(a)

2. Mathematical Preliminaries (cont.)

- An Example of the fluency signal of order 3.



(b)

Figure 1. Waveform of the fluency signal $s(t)$ and the sampling basis. (*a*) Fluency signal spaces. (*b*) An example of the fluency signal of order 3.

2. Mathematical Preliminaries (cont.)

- Mutual relations between : ${}^{m}_{[s]}\varphi$, ${}^{m}_{[b]}\varphi$ and ${}^{m}B$.

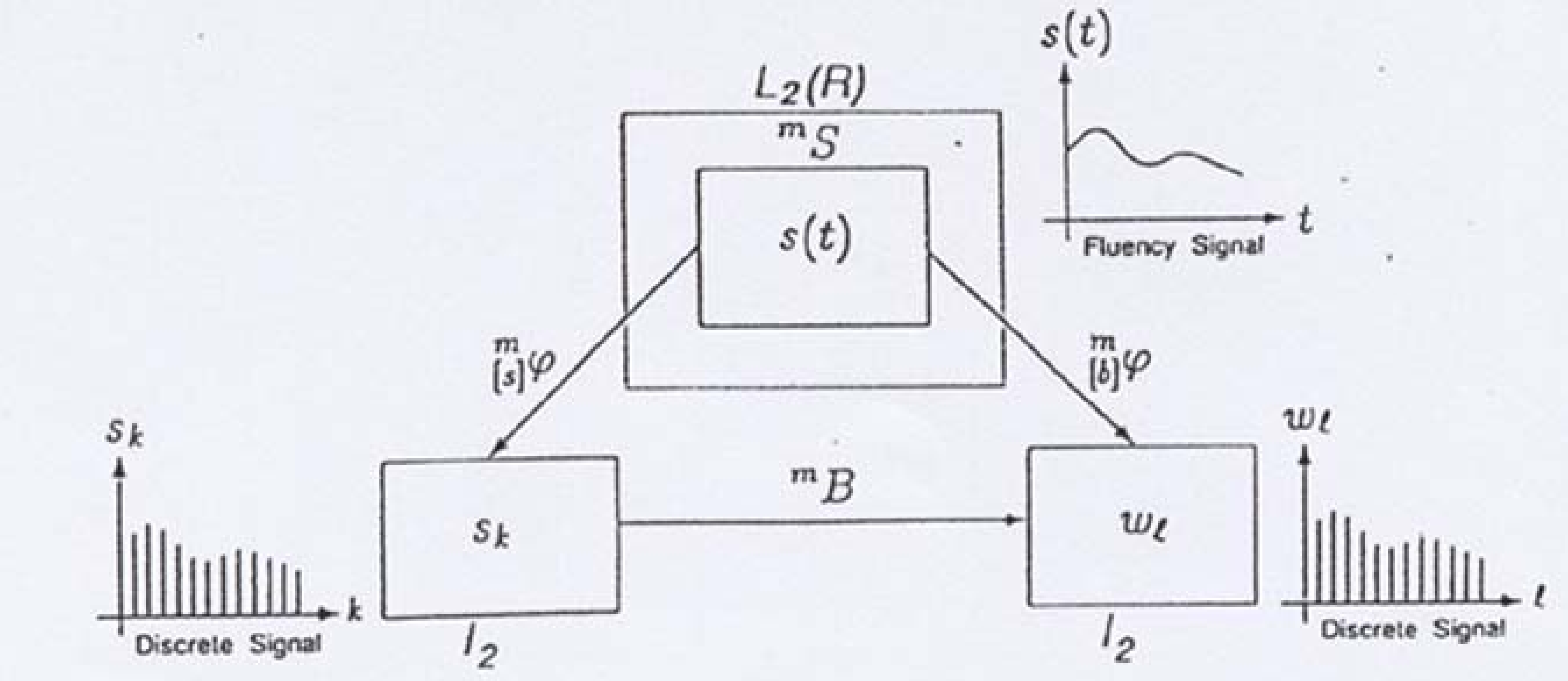


Figure 2. Mutual relations between ${}^{m}_{[s]}\varphi$, ${}^{m}_{[b]}\varphi$ and ${}^{m}B$.

3. Integration of a CT Signal

- Given s_k sample value of CT differential equation

$$s(t) \;=\; \frac{\mathrm{d}x(t)}{\mathrm{d}t} \;=\; \mathrm{f}(x(t), u(t), t)$$

- The problem is to find x_k, the value of $x(t)$ at $t = kh$

$$x_k = \int_{-\infty}^{kh} s(t)\,\mathrm{d}t$$

- The signal $s(t)$ is fluency signal interpolating the vector of the DT sequence s_k by using the m –order fluency sampling basis ${}^{m}_{[s]}\psi_k(t)$

$$s(t) = \sum_{k=-\infty}^{\infty} s_k \, {}^{m}_{[s]}\psi_k(t) \quad \text{where } s(.) \in {}^{m}S, \quad \text{for } m = 1, 2, 3, \ldots$$

- Because of the time – limitted (locally supported) property of the B – spline signal, ${}^{m}_{[s]}\psi_k(t)$ can be replaced by the B – spline basis ${}^{m}_{[b]}\psi_\ell(t)$ with the algorithm that has been proposed by (Kamada, 1988). We can then denote $s(t)$ in the form

$$s(t) = \sum_{\ell=-\infty}^{\infty} w_\ell \, {}^{m}_{[b]}\psi_\ell(t)$$

where w_ℓ is the B – spline coefficient sequence.

3. Integration of a CT Signal (cont.)

- The fluency integration process is as follows.

$$\begin{aligned} x_k &= \int_{-\infty}^{kh} s(t)\,\mathrm{d}t = \int_{-\infty}^{(k-1)h} s(t)\mathrm{d}t + \int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t \\ &= x_{k-1} + \int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t \end{aligned}$$

where $s(t)$ is fluency function of order m, can be represented in the form of B − spline bases of order m and then reordering, we obtain the second term

$$\begin{aligned} \int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t &= \int_{(k-1)h}^{kh} \sum_{\ell=-\infty}^{\infty} w_\ell \, {}_{[b]}^{m}\psi_\ell(t)\,\mathrm{d}t \\ &= \sum_{\ell=-\infty}^{\infty} w_\ell \int_{(k-1)}^{kh} {}_{[b]}^{m}\psi_\ell(t)\,\mathrm{d}t \end{aligned}$$

3. Integration of a CT Signal (cont.)

- Using time – limited (locally supported) property of B – spline signal, then we can rewrite

$$\int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t = \sum_{\ell=\frac{a-(m-1)h}{h}}^{\frac{b-h}{h}} w_\ell \int_{(k-1)h}^{kh} {}_{[b]}^{m}\psi_\ell(t)\,\mathrm{d}t$$

- Integration part can be computed which is representation of B – spline bases of order m in the form of piecewise polynomials of order m as follows

$$\begin{aligned} {}_{[b]}^{m}I_\ell &= \int_{(k-1)h}^{kh} {}_{[b]}^{m}\psi_\ell(t)\,\mathrm{d}t \\ &= \frac{m}{h^{m-1}} \sum_{p=0}^{m} \frac{(-1)^p}{p!(m-p)!} \int_{(k-1)h}^{kh} (t-(\ell+p)h)_+^{m-1}\,\mathrm{d}t \\ &= \frac{m}{h^{m-1}} \sum_{p=0}^{m} \frac{(-1)^p}{p!(m-p)!} \left.\frac{(t-\alpha)_+^m}{m}\right|_{(k-1)h}^{kh} \end{aligned}$$

3. Integration of a CT Signal (cont.)

- We obtain the integration formula, as follows

$$
{}^{m}_{h}Int(t) : \begin{cases} x_k = x_{k-1} + \sum_{\ell=k-m}^{k-1} w_\ell \, {}^{m}_{[b]}I_\ell \\ {}^{m}_{[b]}I_\ell = \frac{m}{h^{m-1}} \sum_{p=0}^{m} \frac{(-1)^p}{p!(m-p)!} \int_{(k-1)h}^{kh} (t-\alpha)_+^{m-1} \, \mathrm{d}t \\ \text{for } \alpha = (\ell + p)h \end{cases}
$$

- That is ${}^{m}_{h}Int(t)$ a fluency integration technique with tunable parameters m (order of fluency approximation) and h (sampling interval) can be choosen according with smoothness property (continuously differentiable) of signal $s(t)$ we deal with.
- Aircraft flight – training simulators are an example of simulations where different nominal flight conditions can require different value of h and m for each flight condition (e.g., flaps up or down, landing gear up or down, and high and low Mach number).

3. Integration of a CT Signal (cont.)

- A slow – varying input signal (e.g., cruise condition of aircraft) will be simulated at slow sampling rate (high h) or low order of fluency approximation (low m).
- Input signal with a high – frequency content (e.g., maneuvering, landing or tracking condition of aircraft) should be simulated at a high sampling rate (low h) or high order of fluency approximation (high m).
- The fluency tunable integration is obtained by selecting the appropriate order m within a given tolerance of the approximation error incurred in assuming the signal, according to the characteristi of the signal of the CT system we are dealing with.

Algorithm of fluency integration

Start

Given : s_k sample value of $s(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \mathrm{f}(x(t), u(t), t)$

Find : $x_k = \int_{-\infty}^{kh} s(t)\,\mathrm{d}t$ value of $x(t)$ at $t = kh$

Assume : $s(t) = \sum_{k=-\infty}^{\infty} s_k\, {}_{[s]}^{m}\psi_k(t) = \sum_{\ell=-\infty}^{\infty} w_\ell\, {}_{[b]}^{m}\psi_\ell(t)$, fluency signal

Init : $h, m, k = 0$ and x_0

Transform sample value s_k to B − spline coefficient w_ℓ : $w_\ell = {}^{m}B\, s_k$

Loop $k = k + 1$

- $a = (k-1)h$ and $b = kh$
- Compute : ${}_{[b]}^{m}I_\ell = \int_a^b {}_{[b]}^{m}\psi_\ell(t)\,\mathrm{d}t$, for ℓ from $\frac{a-(m-1)h}{h}$ to $(\frac{b-h}{h})$
- Compute : $\delta_x = \int_a^b s(t)\,\mathrm{d}t = \sum_{\ell=\frac{a-(m-1)h}{h}=k-m}^{\frac{b-h}{h}=k-1} w_\ell\, {}_{[b]}^{m}I_\ell$
- Compute : $x_k = x_{k-1} + \delta_x$

End

4. Examples for m = 1, 2 and 3

- In the case of order of fluency approximation $m = 1$, $s(t)$ is a staircase signal

$$
\begin{aligned}
x_k &= \int_{-\infty}^{kh} s(t)\,\mathrm{d}t = \int_{-\infty}^{(k-1)h} s(t)\mathrm{d}t + \int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t \\
&= x_{k-1} + \delta_x = x_{k-1} + \sum_{\ell=k-m}^{k-1} w_\ell \, {}^{m}_{[b]}I_\ell \\
&= x_{k-1} + w_{k-1} \sum_{p=0}^{1} \frac{(-1)^p}{p!(1-p)!} \int_{(k-1)h}^{kh} (t-(k-1+p)h)_+^0 \,\mathrm{d}t \\
&= x_{k-1} + w_{k-1} \int_{(k-1)h}^{kh} 1\,\mathrm{d}t \\
&= x_{k-1} + w_{k-1}\,(kh-(k-1)h) = x_{k-1} + hw_{k-1}
\end{aligned}
$$

- This formulation is identical with formula of Euler′s integration method or first order Runge – Kutta method with w_{k-1} equal to sample value of s_{k-1}.

4. Examples for m = 2 (cont.)

- If $m = 2$, then $s(t)$ is a polygonal signal

$$
\begin{aligned}
x_k &= \int_{-\infty}^{kh} s(t)\,\mathrm{d}t = \int_{-\infty}^{(k-1)h} s(t)\mathrm{d}t + \int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t \\
&= x_{k-1} + \delta_x = x_{k-1} + \sum_{\ell=k-m}^{k-1} w_\ell \, {}_{[b]}^{m} I_\ell \\
&= x_{k-1} + \sum_{\ell=k-m}^{k-1} w_\ell \frac{m}{h^{m-1}} \sum_{p=0}^{m} \frac{(-1)^p}{p!(m-p)!} \int\limits_{(k-1)h}^{kh} (t-\alpha)_+^{m-1} \\
&= x_{k-1} + w_{k-2} \frac{2}{h} \frac{h^2}{4} + w_{k-1} \frac{2}{h} \frac{h^2}{4} \\
&= x_{k-1} + \frac{h}{2}(w_{k-2} + w_{k-1})
\end{aligned}
$$

- This formula is like trapezoidal integration method or closed Newton – Cotes integration method.
- The polygonal better approximates the continuous – time signal.

4. Examples for m = 3 (cont.)

- If $m = 3$, then $s(t)$ is a quadratic spline signal

$$\begin{aligned}
x_k &= \int_{-\infty}^{kh} s(t)\,\mathrm{d}t = \int_{-\infty}^{(k-1)h} s(t)\mathrm{d}t + \int_{(k-1)h}^{kh} s(t)\,\mathrm{d}t \\
&= x_{k-1} + \delta_x = x_{k-1} + \sum_{\ell=k-m}^{k-1} w_\ell \, {}_{[b]}^{m} I_\ell \\
&= x_{k-1} + \sum_{\ell=k-3}^{k-1} w_\ell \frac{3}{h^{3-1}} \sum_{p=0}^{3} \frac{(-1)^p}{p!(3-p)!} \int_{(k-1)h}^{kh} (t-\alpha)_+^{3-1}\,\mathrm{d}t \\
&= x_{k-1} + w_{k-3}\frac{3}{h^2}\frac{h^3}{18} + w_{k-2}\frac{3}{h^2}\frac{4h^3}{18} + w_{k-1}\frac{3}{h^2}\frac{h^3}{18} \\
&= x_{k-1} + \frac{h}{6}(w_{k-3} + 4w_{k-2} + w_{k-1})
\end{aligned}$$

- w_{k-2}, w_{k-1} and w_k are B – spline coefficients computed from sample value of $s(t)$.
- The CT signal is naturally modelled as a smooth function.

A series of DT integration formulation

	CT signals	Continuous – Time (CT) integrations
	$s(t) \in L_2(R)$	Given : $s(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \mathrm{f}(x(t), u(t), t)$ Find : $x(kh) = \int_{-\infty}^{kh} s(t)\,\mathrm{d}t$

Order	Assumptions	Discrete – Time (DT) integrations
1	$s(t) \in {}^1S$	$x_k = x_{k-1} + hw_{k-1}$
2	$s(t) \in {}^2S$	$x_k = x_{k-1} + \frac{h}{2}(w_{k-2} + w_{k-1})$
3	$s(t) \in {}^3S$	$x_k = x_{k-1} + \frac{h}{6}(w_{k-3} + 4w_{k-2} + w_{k-1})$
$\cdots$	$\cdots$	$\cdots$
m	$s(t) \in {}^mS$	${}^m_hInt(t)\ :\ x_k = x_{k-1} + \sum_{\ell=k-m}^{k-1} w_\ell\, {}^m_{[b]}I_\ell$ where ${}^m_{[b]}I_\ell = \frac{m}{h^{m-1}} \sum_{p=0}^{m} \frac{(-1)^p}{p!(m-p)!} \int_{(k-1)h}^{kh} (t-\alpha)_+^{m-1}\,\mathrm{d}t$ for $\alpha = (\ell + p)h$

5. Conclusions

- The fluency tunable integration method was derived.
- It included and generalized the conventional one, i.e. Euler′s and Trapezoidal – like integration.
- The tunable parameters m (order of fluency approximation) and h (sampling interval) can be choosen adaptivelly according with smoothness (continuously differentiable) of signal.
- This concept provided a better relationship between the discrete – time (DT) and continonuous – time (CT) signal.

6. Further Investigation

- Because the discrete signal (w_k) is the B – Spline coefficient computed from sampled signal (s_k), such computation time leads to some problems in the real – time application.
- Further interesting research is to apply this method to the real – time discrete – time system field, for example : real – time digital flight simulation.

7. References

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Thanks for Your Attention